Pomeron Physics and QCD

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Contents

1

Properties of the S-matrix

In this chapter we specify the kinematics, define the normalisation of amplitudes and cross sections and establish the basic formalism used throughout. All mathematical functions used, and their properties, can be found in [9].

1.1 Kinematics

We consider first the two-body scattering process $1 + 2 \to 3 + 4$ of figure 1.1, where the particles have masses m_i and four-momenta P_i, $i = 1, \ldots, 4$. Our notation is that the four-momentum of a particle is $P = (E, \mathbf{p})$, where E is its energy and $\mathbf{p}$ its three-momentum, and we write

$$P_1.P_2 = E_1E_2 - \mathbf{p}_1.\mathbf{p}_2. \tag{1.1}$$

The Lorentz-invariant variables s, t and u, called Mandelstam variables, are defined by

$$\begin{aligned} s &= (P_1 + P_2)^2 \\ t &= (P_1 - P_3)^2 \\ u &= (P_1 - P_4)^2 \end{aligned} \tag{1.2}$$

with the relation

$$s + t + u = \sum_{i=1}^{4} m_i^2 \,. \tag{1.3}$$

Equation (1.3) means that a two-body amplitude is a function of only two independent variables. We shall normally take these to be s and t, with u defined via (1.3), and write the amplitude as $A(s, t)$. However, sometimes

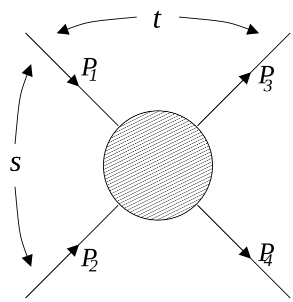

Figure 1.1. Two-body scattering process $1 + 2 \to 3 + 4$

it will be more appropriate to use s and u, or t and u, as the independent variables, and then write the amplitude as $A(s, u)$ or $A(t, u)$.

Figure 1.1 not only describes the scattering process $1 + 2 \to 3 + 4$ in the s-channel but, by reversing the signs of some of the four-momenta, it can also represent the t-channel process $1 + \bar{3} \to \bar{2} + 4$ and the u-channel process $1 + \bar{4} \to 3 + \bar{2}$, where the bar denotes the antiparticle.

In the s-channel centre-of-mass frame of the initial particles 1 and 2, the four-momenta are given explicitly by

$$\begin{aligned} P_1 &= (E_1, \mathbf{p}_1) \qquad & P_2 &= (E_2, -\mathbf{p}_1) \\ P_3 &= (E_3, \mathbf{p}_3) \qquad & P_4 &= (E_4, -\mathbf{p}_3) \end{aligned} \tag{1.4}$$

where E_i is the energy of particle i, $\mathbf{p}_1$ is the three-momentum of particle 1 and $\mathbf{p}_3$ the three-momentum of particle 3 in this frame. Then

$$s = (E_1 + E_2)^2 = (E_3 + E_4)^2 \tag{1.5}$$

and

$$\begin{aligned} E_1 &= \frac{1}{2\sqrt{s}}(s + m_1^2 - m_2^2) \quad & E_2 &= \frac{1}{2\sqrt{s}}(s + m_2^2 - m_1^2) \\ E_3 &= \frac{1}{2\sqrt{s}}(s + m_3^2 - m_4^2) \quad & E_4 &= \frac{1}{2\sqrt{s}}(s + m_4^2 - m_3^2) \end{aligned} \tag{1.6}$$

and

$$\begin{aligned} \mathbf{p}_1^2 &= \frac{1}{4s}[s - (m_1 + m_2)^2]\,[s - (m_1 - m_2)^2] \\ \mathbf{p}_3^2 &= \frac{1}{4s}[s - (m_3 + m_4)^2]\,[s - (m_3 - m_4)^2]. \end{aligned} \tag{1.7}$$

From (1.2) and (1.4),

$$\begin{aligned} t &= m_1^2 + m_3^2 - 2(E_1E_3 - \mathbf{p}_1.\mathbf{p}_3) \\ &= m_1^2 + m_3^2 - 2(E_1E_3 - |\mathbf{p}_1||\mathbf{p}_3|\cos\theta_s) \\ u &= m_1^2 + m_4^2 - 2(E_1E_4 + \mathbf{p}_1.\mathbf{p}_3) \\ &= m_1^2 + m_4^2 - 2(E_1E_4 + |\mathbf{p}_1||\mathbf{p}_3|\cos\theta_s) \end{aligned} \tag{1.8}$$

where θ_s is the angle between the three-momenta of particles 1 and 3 in the s-channel centre-of-mass frame, that is it is the centre-of-mass-frame scattering angle.

The physical region for the s-channel is given by

$$s \geq (m_1 + m_2)^2 \qquad \text{and} \qquad -1 \leq \cos\theta_s \leq 1. \tag{1.9}$$

For arbitrary masses the boundary of the physical region as a function of s and t is rather complicated. It is simpler for equal masses $m_i = m$, $i = 1, \ldots, 4$, so that $\mathbf{p}_1 = \mathbf{p}_3 = \mathbf{p}$ and

$$\begin{aligned} s &= 4(\mathbf{p}^2 + m^2) \\ t &= -2\mathbf{p}^2(1 - \cos\theta_s) \\ u &= -2\mathbf{p}^2(1 + \cos\theta_s). \end{aligned} \tag{1.10}$$

The physical region for s-channel scattering is then given by $s \geq 4m^2$, $t \leq 0$ and $u \leq 0$. In this channel, s is an energy squared and each of t and u is a momentum transfer squared. Similarly the physical region for t-channel scattering is $t \geq 4m^2$, $u \leq 0$, $s \leq 0$; and for u-channel scattering it is $u \geq 4m^2$, $s \leq 0$, $t \leq 0$. The symmetry between s, t and u is readily demonstrated by plotting the physical regions in the s-t plane with the s and t axes inclined at 60°, as shown in figure 1.2.

1.2 The cross section

For orthonormal states $\langle f|$ and $|i\rangle$, that satisfy $\langle f|f\rangle = \langle i|i\rangle$ and $\langle f|f'\rangle = \delta_{ff'}$, the S-matrix element $\langle f|S|i\rangle$ is defined such that

$$P_{fi} = |\langle f|S|i\rangle|^2 = \langle i|S^\dagger|f\rangle\langle f|S|i\rangle \tag{1.11}$$

is the probability of $|f\rangle$ being the final state, given $|i\rangle$ as the initial state. If the set of orthonormal states $|f\rangle$ is complete,

$$\sum_f |f\rangle\langle f| = 1. \tag{1.12}$$

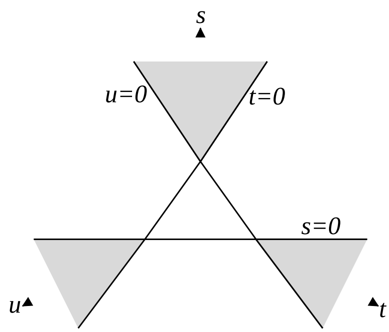

Figure 1.2. Physical regions for equal-mass scattering such as $\pi\pi \to \pi\pi$

Starting from the initial state $|i\rangle$, the probability of ending up in some final state must be unity so

$$1 = \sum_f |\langle f|S|i\rangle|^2 = \sum_f \langle i|S^\dagger|f\rangle\langle f|S|i\rangle = \langle i|S^\dagger S|i\rangle. \qquad (1.13)$$

Since (1.13) must be true for any choice of the complete set of basis states $|i\rangle$ it follows that $S^\dagger S = 1$. Similarly the requirement that any final state $|f\rangle$ has originated from some initial state $|i\rangle$ yields $SS^\dagger = 1$. That is, S is unitary.

We now go over to the case of continuum states and specialise to a two-body initial state. The scattering matrix S is related to the transition matrix T by

$$\langle f|S|i\rangle = \langle P_1' P_2' \ldots P_n'|S|P_1 P_2\rangle = \delta_{fi} + i(2\pi)^4\delta^4(P^f - P^i)\,\langle f|T|i\rangle \qquad (1.14)$$

where P^i is the sum of the initial four-momenta and P^f the sum of the final four-momenta. The scattering amplitude is normalised such that the transition rate per unit time per unit volume from the initial state $|i\rangle = |P_1 P_2\rangle$ to the final state $|f\rangle = |P_1' \cdots P_n'\rangle$ is

$$R_{fi} = (2\pi)^4\delta^4(P^f - P^i)\,|\langle f|T|i\rangle|^2. \qquad (1.15)$$

The total cross section for the reaction $12 \to n$ particles is

$$\sigma_{12\to n} = \frac{1}{4|\mathbf{p}_1|\sqrt{s}} \sum (2\pi)^4\delta^4(P^f - P^i)\,|\langle f_n|T|i\rangle|^2 \qquad (1.16)$$

where the sum is over the momenta of the particles in the n-particle state $\langle f_n|$. That is, with $\delta^+(p^2 - m^2) = \delta(p^2 - m^2)\,\theta(p^0)$,

$$\sigma_{12\to n} = \frac{1}{4|\mathbf{p}_1|\sqrt{s}} \int \left(\prod_{i=1}^{n} \frac{d^4 P_i'}{(2\pi)^4} \, 2\pi\delta^+ (P_i'^2 - m_i^2) \right)$$
$$\times (2\pi)^4 \delta^4 \Big(\sum_{i=1}^{n} P_i' - P_1 - P_2 \Big) \, |\langle P_1' \cdots P_n' | T | P_1 P_2 \rangle|^2$$
$$= \frac{1}{4|\mathbf{p}_1|\sqrt{s}} \int \left(\prod_{i=1}^{n} \frac{d^3 p_i'}{2E_i (2\pi)^3} \right) (2\pi)^4 \delta^4 \Big(\sum_{i=1}^{n} P_i' - P_1 - P_2 \Big)$$
$$\times |\langle P_1' \cdots P_n' | T | P_1 P_2 \rangle|^2. \tag{1.17}$$

Here, $\mathbf{p}_1$ is the initial momentum in the s-channel centre-of-mass frame. It is given by (1.7):

$$|\mathbf{p}_1|^2 s = (P_1.P_2)^2 - m_1^2 m_2^2 = \tfrac{1}{4} [s - (m_1 + m_2)^2] \, [s - (m_1 - m_2)^2]. \tag{1.18}$$

We must use this in (1.17), which then gives the cross section in any frame: it is Lorentz invariant, and the momentum integrations may be performed in any frame.

We may calculate a differential cross section $d\sigma_{12\to n}/d\omega$. Typically, ω will be a momentum transfer between an initial and a final particle, or the corresponding scattering angle, or the energy of one of the final particles. To calculate the differential cross section, we first express ω as a function $\omega(P_i, P_f')$ of the various momenta, and then include $\delta(\omega - \omega(P_i, P_f'))$ in the integrations in (1.17). For example, when the final state contains just two particles and t is the momentum transfer defined in (1.2),

$$\frac{d\sigma_{12\to 34}}{dt} = \frac{1}{4|\mathbf{p}_1|\sqrt{s}} \int \frac{d^4 P_3}{(2\pi)^4} 2\pi\delta^+ ({P_3}^2 - m_3^2) \, \frac{d^4 P_4}{(2\pi)^4} 2\pi\delta^+ ({P_4}^2 - m_4^2)$$
$$\times (2\pi)^4 \delta^4 (P_1 + P_2 - P_3 - P_4) |\langle P_3 P_4 | T | P_1 P_2 \rangle|^2 \delta(t - (P_1 - P_3)^2)$$
$$= \frac{1}{64\pi |\mathbf{p}_1|^2 s} |\langle P_3 P_4 | T | P_1 P_2 \rangle|^2 \, \delta(t - (P_1 - P_3)^2). \tag{1.19}$$

In the equal-mass case this gives

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s (s - 4m^2)} \, |\langle P_3 P_4 | T | P_1 P_2 \rangle|^2. \tag{1.20}$$

The formulae in this section apply when the particles involved have no spin or, if they do have spin, when we average over initial spin states and sum over final spin states.

1.3 Unitarity and the optical theorem

Unitarity provides an important connection between the total cross section and the forward ($\theta_s = 0$) elastic scattering amplitude; this connection is known as the optical theorem. Because the operator S is unitary, so that $SS^\dagger = 1$, for any orthonormal states $\langle j|$ and $|i\rangle$

$$\delta_{ji} = \langle j|SS^\dagger)|i\rangle = \sum_f \langle j|S|f\rangle\langle f|S^\dagger)|i\rangle \tag{1.21}$$

where we have used the completeness relation (1.12). With the definition (1.14) of the T-matrix, this is

$$\langle j|T|i\rangle - \langle j|T^\dagger|i\rangle = (2\pi)^4 i \sum_f \delta^4(P^f - P^i)\langle j|T^\dagger|f\rangle\langle f|T|i\rangle. \tag{1.22}$$

For the particular case $j = i$,

$$2\,\mathrm{Im}\,\langle i|T|i\rangle = \sum_f (2\pi)^4 \delta^4(P^f - P^i)|\langle f|T|i\rangle|^2. \tag{1.23}$$

The right-hand side is (1.15) summed over f: it is the total transition rate. This gives us the total cross section, which is (1.17) summed over n, the number of final-state particles:

$$\sigma_{12}^{\mathrm{Tot}} = \frac{1}{2|\mathbf{p}_1|\sqrt{s}}\,\mathrm{Im}\,\langle i|T|i\rangle. \tag{1.24}$$

Here, $|\mathbf{p}_1|$ is again the magnitude of the initial centre-of-mass frame three-momentum, which is given by (1.18). $\langle i|T|i\rangle$ is the scattering amplitude for the reaction $1 + 2 \to 1 + 2$ with the direction of motion of the particles unchanged, that is it is the forward scattering amplitude, $\theta_s = 0$. For $m_3 = m_1$ and $m_4 = m_2$ the forward direction corresponds to $t = 0$. Then

$$\sigma_{12}^{\mathrm{Tot}} = \frac{1}{2|\mathbf{p}_1|\sqrt{s}}\,\mathrm{Im}\,A(s, t = 0) \tag{1.25}$$

where $A(s, t)$ is the elastic scattering amplitude. Equation (1.24) or (1.25) is the optical theorem.

1.4 Crossing and analyticity

The basic principle of crossing is that the same function $A(s, t)$ analytically continued to the three physical regions of figure 1.2 gives the corresponding scattering amplitude there, with s, t, u related by (1.3). This is obviously

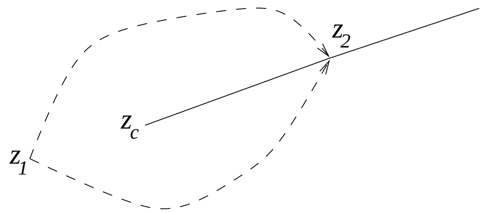

Figure 1.3. Paths of analytic continuation that pass round different sides of a branch point

true order by order for Feynman diagrams. For example Coulomb scattering ($e^-e^- \to e^-e^-$) and Bhabha scattering ($e^+e^- \to e^+e^-$) are described by the same Feynman diagrams.

It is necessary to make some assumption about the analytic structure of the scattering amplitude $A(s,t)$ in order to continue from one region to another. The assumption usually made is that any singularity has a dynamical origin. Poles are associated with bound states and thresholds give rise to cuts. For example in the s-plane a bound state of mass $m_B = \sqrt{s_B}$ will give rise to a pole at $s = s_B$ and there will be cuts with branch points corresponding to physical thresholds. These arise because of the unitarity condition (1.23). In this condition, $P^{f2} = s$ is the squared invariant mass of the state f, which shows that n-particle states contribute to the imaginary part of the amplitude if $\sqrt{s}$ is greater than the n-particle threshold energy. The threshold for producing a state in which the particles have masses $M_1, M_2, M_3, \ldots$ is at $s = (M_1 + M_2 + M_3 + \cdots)^2$. In a model with only one type of particle, of mass m, the thresholds are at $s = 4m^2, 9m^2, \ldots$. Each corresponds to a branch point of $A(s,t)$. When a function $f(z)$ of a complex variable z has a branch point at some point z_c, we attach a cut to the branch point, to remind us that continuing $f(z)$ from z_1 to z_2 along paths that pass to different sides of the branch point results in different values for the function: see figure 1.3. We say that $f(z)$ has a discontinuity across the cut. Since we may choose the point z_2 to lie in any direction relative to z_1, we must be prepared to draw the cut in any direction. It need not be a straight line. The only constraint is that one end of it is at $z = z_c$ and does not cross any other singularity. For $A(s,t)$, therefore, we need a cut attached to each branch point $s = 4m^2, 9m^2, 16m^2, \ldots$. By convention, we draw each cut along the real axis, so that the one attached to $s = 4m^2$ passes through all the other branch points and effectively all these branch points need only one cut, the right-hand one in figure 1.4.

A consequence of the assumption of analyticity is crossing symmetry. Con-

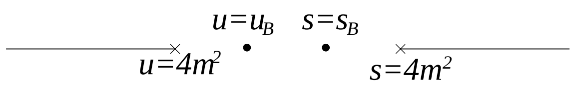

Figure 1.4. Poles and cuts in the complex s-plane for equal mass scattering for a given, fixed t. Recall that $u = 4m^2 - s - t$.

sider the scattering process

$$a + b \to c + d \tag{1.26}$$

and write its amplitude as $A_{a+b\to c+d}(s,t,u)$, reinstating the variable u for symmetry, but remembering that it is not independent being given in terms of s and t by (1.3). The physical region for the process (1.26) is $s > \max\{(m_a + m_b)^2, (m_c + m_d)^2\}$. In the equal-mass case, $t, u < 0$; in the unequal-mass case the constraint on t and u is more complicated, but most of the physical region lies in $t, u < 0$. The amplitude may be continued analytically to the region $t > \max\{(m_a + m_{\bar{c}})^2, (m_{\bar{b}} + m_d)^2\}$ and $s, u < 0$. This gives the amplitude for the t-channel process

$$a + \bar{c} \to \bar{b} + d \tag{1.27}$$

where $\bar{b}$ and $\bar{c}$ mean respectively the antiparticles of b and c. That is, we have

$$A_{a+\bar{c}\to\bar{b}+d}(t,s,u) = A_{a+b\to c+d}(s,t,u). \tag{1.28}$$

Similarly for the u-channel process

$$a + \bar{d} \to \bar{b} + d \tag{1.29}$$

we have

$$A_{a+\bar{d}\to\bar{b}+c}(u,t,s) = A_{a+b\to c+d}(s,t,u). \tag{1.30}$$

There are various mathematical results about the analytic properties of scattering amplitudes. Although these results are not complete, what is known is consistent with the assumption that the analytic structure in the complex s-plane for equal mass scattering is that shown in figure 1.4. The right-hand cut, from $s = 4m^2$ to ∞, arises from the physical thresholds in the s-channel. The pole at $s = s_B$ assumes that there is a bound state in the s-channel with mass $m_B = \sqrt{s_B}$. The left hand cut and pole arise respectively from the physical thresholds in the u-channel and an assumed u-channel bound state at $u = u_B$. The position of the singularities in the s-plane arising from u-channel effects is given by the relation (1.3). Thus the presence of a threshold at $u = u_0$ for positive u means that the amplitude $A(s,t)$ must have a cut along the negative real axis with a branch point at $s = \bar{s}_0 = 4m^2 - t - u_0$, so that $\bar{s}_0 = -t$ when $u_0 = 4m^2$. Equally, a bound-state pole at $u = u_B$ will give rise to a pole at $s = 4m^2 - t - u_B$. In figure 1.4 we have drawn the u-channel bound-state pole and the u-channel cut to the left of the corresponding s-channel singularities. However, they move as t varies and for physical values of t, $t \leq 0$, the u-channel pole is actually to the right of the s-channel pole, and when t is sufficiently large negative the two cuts actually overlap.

In perturbation theory, masses are assigned a small negative imaginary part, $m^2 \to m^2 - i\epsilon$, which is made to go to zero at the end of any calculation. The same $i\epsilon$ prescription is used outside the framework of perturbation theory; for example it makes Minkowski-space path integrals converge for large values of the fields. In figure 1.4, the $i\epsilon$ prescription pushes the branch point at $s = 4m^2$ downwards in the complex s-plane, and likewise the branch points corresponding to the higher thresholds, $s = 9m^2, s = 16m^2, \ldots$. As $\epsilon \to 0$, the branch points move back on to the real axis from below. That is, the physical s-channel amplitude is reached by analytic continuation down on to the real axis from the upper half of the complex s-plane. This is equivalent to saying that the physical amplitude is

$$\lim_{\epsilon \to 0} A(s + i\epsilon, t). \tag{1.31}$$

If we analytically continue it to real values of s between s_B and $4m^2$, there is no cut and the amplitude is real there[10]. The Schwarz reflection principle tells us that an analytic function $f(s)$ which is real for some range of real values of s satisfies

$$f(s^*) = [f(s)]^*.$$

So if we make a further continuation via the lower half of the complex plane, back to real values of s greater than $4m^2$, we obtain the complex conjugate of the physical amplitude:

$$A(s - i\epsilon, t) = [A(s + i\epsilon, t)]^*. \tag{1.32}$$

Therefore, for $s \geq 4m^2$ and $-s < t, u \leq 0$,

$$2i \,\mathrm{Im}\, A(s + i\epsilon, t) = A(s + i\epsilon, t) - A(s - i\epsilon, t) \tag{1.33}$$

where it is understood in this equation that we have to take the limit $\epsilon \to 0$. (By convention the imaginary part of the amplitude is defined to be real, as is evident from the factor $2i$.) The right hand side of (1.33) is called the s-channel discontinuity, denoted by $D_s(s,t,u)$.

Similar arguments can be applied to the physical t-channel and u-channel processes $1+\bar{3} \to \bar{2}+4$ and $1+\bar{4} \to 3+\bar{2}$. Thus there must be cuts along the

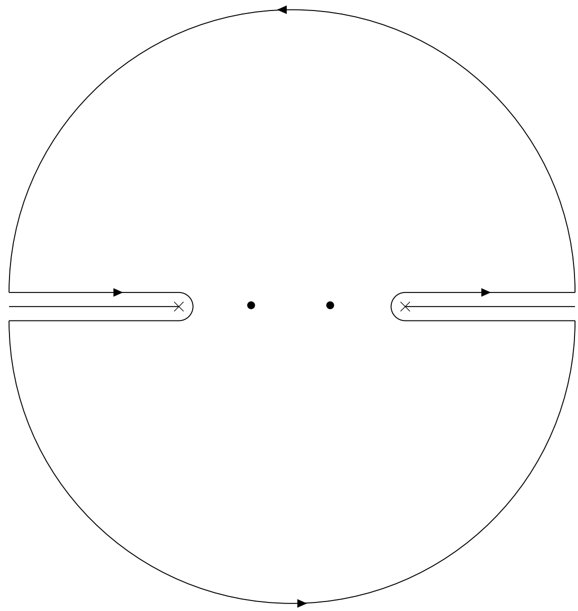

Figure 1.5. Contour of integration in the complex s'-plane

real positive t and u axes, with branch points at the appropriate physical thresholds in these channels, and possibly poles as well. Equivalently to (1.33) we define the t-channel and u-channel discontinuities by

$$\begin{aligned} D_t(s,t,u) &= A(s,t+i\epsilon) - A(s,t-i\epsilon) = 2i\,\mathrm{Im}\,A(s,t+i\epsilon) \\ &\qquad\qquad t > 4m^2 \text{ and } u,s \le 0 \\ D_u(s,t,u) &= A(s,u+i\epsilon) - A(s,u-i\epsilon) = 2i\,\mathrm{Im}\,A(s,u+i\epsilon) \\ &\qquad\qquad u > 4m^2 \text{ and } s,t \le 0 \end{aligned} \tag{1.34}$$

where again the limit $\epsilon \to 0$ is understood.

Knowing the analytic structure of an amplitude allows us to derive a "dispersion relation". We fix t and use the contour of integration shown in figure 1.5, which must be such that the point $s = s'$ is within it. Then $(s'-s)^{-1}A(s',t)$ is analytic within the contour except for a pole at $s' = s$, so that Cauchy's theorem tells us that the integral of this function is just the residue at the pole, which is $2\pi i A(s,t)$. Hence

$$A(s,t) = \frac{1}{2\pi i}\oint ds' \frac{A(s',t)}{s'-s} \tag{1.35}$$

with u (u') given in terms of s (s') and t by (1.3). Assume for the moment

Figure 1.6. A contour of integration equivalent to that of figure 1.5

that $A(s,t)$ goes to zero like some negative power of s as $|s| \to \infty$, so that the contribution from the circle at infinity will vanish. So instead of the contour of figure 1.5 we may use the two-piece contour of figure 1.6. For the integration along each piece, we pick up the residue at the bound-state pole within the new contour, together with integrals in opposite directions along the upper and lower sides of the cut, which is just the integral of the discontinuity across the cut:

$$A(s,t) = \frac{g_s^2}{s - s_B} + \frac{g_u^2}{u - u_B} + \frac{1}{2\pi i}\int_{s_0}^{\infty} ds' \frac{D_s(s',t,u')}{s'-s} + \frac{1}{2\pi i}\int_{u_0}^{\infty} du' \frac{D_u(s',t,u')}{u'-u}. \tag{1.36}$$

Here s_0, u_0 are the thresholds of the lowest states accessible to that channel. For example in nucleon-nucleon scattering $s_0 = 4m_N^2$ and $u_0 = 4m_\pi^2$.

We may write the dispersion relation (1.36) more compactly if we extend the definition of the discontinuities $D(s,t,u)$ to include any bound-state contributions. So we define

$$D_s(s,t,u) = -2\pi i g_s^2 \delta(s - s_B) \qquad s < 4m^2 \tag{1.37}$$

with similar definitions for $D_t(s,t,u)$ and $D_u(s,t,u)$. Then (1.36) simplifies to

$$A(s,t) = \frac{1}{2\pi i}\int_0^{\infty} ds' \frac{D_s(s',t,u')}{s'-s} + \frac{1}{2\pi i}\int_0^{\infty} du' \frac{D_u(s',t,u')}{u'-u}. \tag{1.38}$$

The denominator of the first integral vanishes for $s' = s$ and for physical values of s the s' integration passes through this value. We recall from (1.31) that we must give s a small positive imaginary part $i\epsilon$ to obtain the physical amplitude. This prevents the denominator from vanishing. We may write the resulting first denominator as

$$\frac{1}{s' - s - i\epsilon} = P\frac{1}{s'-s} + i\pi\delta(s'-s) \tag{1.39}$$

where P denotes "principal value". The denominator of the second integral does not vanish and this term is real, so (1.38) is equivalent to

$$\text{Re } A(s,t) = \frac{1}{2\pi i} P\int_0^{\infty} ds' \frac{D_s(s',t,u')}{s'-s} + \frac{1}{2\pi i}\int_0^{\infty} du' \frac{D_u(s',t,u')}{u'-u}. \tag{1.40}$$

Equally we can write fixed-s or fixed-u dispersion relations. For example, a fixed-s dispersion relation has the form

$$A(t,u) = \frac{1}{2\pi i}\int_0^\infty dt'\,\frac{D_t(s,t',u')}{t'-t} + \frac{1}{2\pi i}\int_0^\infty du'\,\frac{D_u(s,t',u')}{u'-u}. \tag{1.41}$$

If the amplitude does not go to zero sufficiently quickly as $|s| \to \infty$ for the contribution from the circle at infinity to vanish in the integral (1.35), then choose some value s_1 of s and write (1.35) as

$$\begin{aligned} A(s,t) - A(s_1,t) &= \frac{1}{2\pi i}\oint ds'\,A(s',t)\Big(\frac{1}{s'-s} - \frac{1}{s'-s_1}\Big) \\ &= \frac{1}{2\pi i}(s-s_1)\oint ds'\,\frac{A(s',t)}{(s'-s)(s'-s_1)}. \end{aligned} \tag{1.42}$$

Then if $s^{-1}A(s,t)$ goes to zero like some negative power of s, the integral over the infinite-circular part of the contour in figure 1.5 again vanishes, and we can manipulate as before. The resulting dispersion relation is called a once-subtracted dispersion relation. If one subtraction is not enough for us to be able to discard the contribution from the circular part of the contour, we may make another:

$$\begin{aligned} &A(s,t) - A(s_1,t) - (s-s_1)\frac{\partial}{\partial s_1}A(s_1,t) = \\ &\qquad\qquad \frac{1}{2\pi i}(s-s_1)^2\oint ds'\,\frac{A(s',t)}{(s'-s)(s'-s_1)^2} \end{aligned} \tag{1.43}$$

and so on.

A particularly useful form of dispersion relation is for forward elastic scattering, such as $NN \to NN$ at $t=0$, as the optical theorem (1.25) allows us to obtain the imaginary part of the amplitude from the total cross section.

1.5 Partial-wave amplitudes

According to (1.7) and (1.8), at fixed s the momentum transfer t varies linearly with

$$z_s = \cos\theta_s \tag{1.44}$$

where θ_s is the s-channel scattering angle in the centre-of-mass frame. Hence, instead of s and t as the independent variables we can use s and z_s, that is $t = t(s, z_s)$. Similarly $u = u(s, z_s)$. The amplitude $A(s,t)$ can then be written as $A(s, t(s, z_s))$, and expanded in the partial-wave series

$$A(s, t(s, z_s)) = 16\pi \sum_{l=0}^{\infty} (2l+1) A_l(s) P_l(z_s) \tag{1.45}$$

where $P_l(z)$ is the Legendre polynomial of the first kind, of order l. We shall discuss later the modifications required for the inclusion of spin. The factor 16π is included so that in the nonrelativistic limit the partial-wave amplitude $A_l(s)$ has the conventional normalisation. Just as in nonrelativistic scattering, it can be written in terms of a real phase shift δ_l and an inelasticity η_l:

$$A_l = \frac{\eta_l(s) e^{2i\delta_l(s)} - 1}{2i\rho(s)} \tag{1.46}$$

where $\rho(s) = 2|\mathbf{p}_1|/\sqrt{s}$ with our choice of normalisation. Below the first inelastic threshold $\eta_l = 1$ and (1.46) can be written as

$$A_l = \frac{e^{i\delta_l} \sin \delta_l}{\rho(s)}. \tag{1.47}$$

Otherwise unitarity requires that $0 \leq \eta_l \leq 1$. We often choose to make δ_l complex by adding to it an imaginary part such that $\eta_l = e^{-2\,\mathrm{Im}\,\delta_l}$; then (1.46) becomes

$$A_l = \frac{e^{2i\delta_l} - 1}{2i\rho(s)}. \tag{1.48}$$

The condition $0 \leq \eta_l \leq 1$ corresponds to

$$\mathrm{Im}\ \delta_l \geq 0. \tag{1.49}$$

Because of the orthogonality of the Legendre polynomials

$$\int_{-1}^{+1} dz\ P_l(z) P_m(z) = \frac{2}{2l+1} \delta_{lm} \tag{1.50}$$

(1.45) can be inverted to give

$$A_l(s) = \frac{1}{32\pi} \int_{-1}^{+1} dz_s\ P_l(z_s) A(s, t(s, z_s)). \tag{1.51}$$

1.6 The Froissart-Gribov formula

The partial-wave series for $A(s,t)$ cannot converge for all s and t, as $P_l(z)$ for integer $l \geq 0$ is an entire function of z_s so $A(s,t)$, as defined by (1.45), would have no singularities in t (or u). The series must diverge at the nearest t (or u) singularity. According to (1.6), (1.7) and (1.8), for fixed physical values of s both t and u depend linearly on z_s and the t-channel poles and thresholds occur for values of z_s greater than 1, while the u-channel poles and thresholds occur for values less than -1. That is, in the complex z_s-plane the right-hand singularities correspond to the t-channel singularities and the left-hand singularities correspond to the u-channel singularities. It may be shown[11] that the partial-wave series (1.45) converges for values of z_s within the Lehmann ellipse, which is an ellipse in the complex z_s-plane with foci at $z_s = \pm 1$ and passing through the nearest singularity.

We can derive an alternative expression for the partial-wave amplitudes which incorporates some information about the analytic structure of $A(s,t)$. It is particularly useful for determining the behaviour of the partial-wave amplitudes for large values of l. It also provides the basis for the continuation of the partial-wave amplitudes to complex values of l needed for Regge theory. This is explained in the next chapter and in appendix A. The alternative form for $A_l(s)$ is known as the the Froissart-Gribov formula. It makes use of the Legendre functions of the second kind, $Q_l(z)$. These have branch points at $z = \pm 1$ and for physical values of l, that is $l = 0, 1, 2, \ldots$, one may choose to draw the associated cuts as a single cut along the real axis joining these two points. The discontinuity of $Q_l(z)$ across this cut is

$$Q_l(z + i\epsilon) - Q_l(z - i\epsilon) = -i\pi P_l(z) \qquad (|z| \le 1). \tag{1.52}$$

We use this to replace $P_l(z_s)$ in the integral (1.51) that defines $A_l(s)$. Then the integral of the $Q_l(z + i\epsilon)$ term corresponds to an integral over z_s along the upper side of the cut, and the integral of the $Q_l(z - i\epsilon)$ term corresponds to one along the lower side of the cut, in the opposite direction because of the minus sign in (1.52). That is,

$$A_l(s) = \frac{i}{32\pi^2} \oint dz_s \, Q_l(z_s) A(s, t(s, z_s)) \tag{1.53}$$

where the integral is round the contour shown in figure 1.7a. In this figure we have drawn also on the right-hand real z_s-axis the t-channel cut and a t-channel bound-state pole (if there is one), with the u-channel cut and pole on the left-hand real axis.

Apart from these singularities, and the two branch points of $Q_l(z)$, the integrand of (1.53) is analytic, and so by Cauchy's theorem the integral of the same integrand round the closed contours in figure 1.7b vanishes. If the integrand vanishes rapidly enough as $z_s \to \infty$, the contributions to this integral from the infinite semicircles vanishes. Up to an l-dependent factor, for large z the behaviour of $Q_l(z)$ is z^{-l}. Hence it is valid to discard the

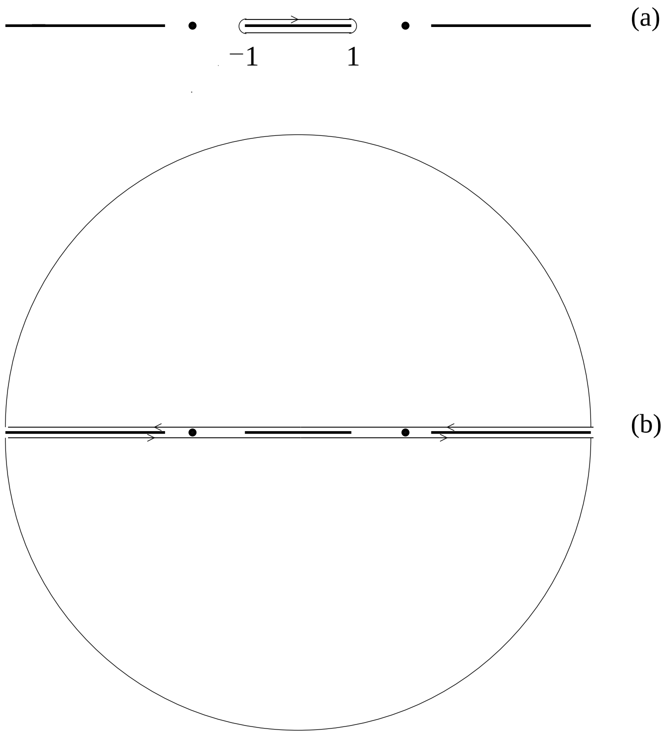

Figure 1.7. Contours of integration in the z_s-plane

semicircular parts of the contour for large enough l, leaving the parts of the contour along the real axis. The integral along the right-hand part is just the integral of $Q_l(z_s)$ times the discontinuity D_t of $A(s,t)$ across the t-channel singularities, defined in (1.34), while the integral along the left-hand part similarly involves the integral of D_u. As in (1.37), we extend the definitions of these discontinuities to the gaps between the branch points, where the bound-state poles lie. So, for large enough l,

$$32\pi^2 i A_l(s) = \int_1^\infty dz'_s \; D_t(s, t(s, z'_s))\, Q_l(z'_s) + \int_{-1}^{-\infty} dz'_s \; D_u(s, u(s, z'_s))\, Q_l(z'_s). \qquad (1.54)$$

This is the Froissart-Gribov formula.

1.7 The Froissart bound

In this section, we explain why the asymptotic ($s \to \infty$) behaviour of a scattering amplitude is limited by s-channel unitarity and the finite range of the forces.

To see this, start with the Froissart-Gribov formula (1.54) for s-channel partial waves. When $z > 1$, the behaviour of $Q_l(z_t')$ for large l is given, up to a constant factor, by[9]

$$Q_l(z) \sim \frac{1}{l^{\frac{1}{2}}(z^2-1)^{\frac{1}{4}}} e^{-(l+\frac{1}{2})\log(z+\sqrt{z^2-1})}. \tag{1.55}$$

When $z < -1$, we may use the relation

$$Q_l(-z) = (-1)^l Q_l(z) \tag{1.56}$$

to deduce the large-l behaviour. Hence at fixed s, the large-l behaviour of $A_l(s)$ is controlled by the value z_0 of the singularity of $A(s, z_s(s,t))$ nearest to the origin in the z_s-plane. Usually this singularity is a t-channel or u-channel bound-state pole. In the neighbourhood of such a t-channel pole,

$$D_t(s,t,u) = -2\pi i g_t^2 \delta(t - t_B) \tag{1.57}$$

as in (1.37). We use (1.10), so that (1.57) is equivalent to

$$D_t(s, t(s, z_s')) \sim 4\pi i g_t^2 |\mathbf{p}|^{-2} \delta(z_s' - z_0). \tag{1.58}$$

The expression for a u-channel bound state is exactly similar. So for large l

$$A_l(s) \sim |\mathbf{p}|^{-2} l^{-\frac{1}{2}} e^{-(l+\frac{1}{2})\zeta(z_0)}. \tag{1.59}$$

The value of z_s at the singularity is

$$z_0 \sim 1 + t_B/2\mathbf{p}^2 \quad \text{or} \quad -1 - u_B/2\mathbf{p}^2 \tag{1.60}$$

so that

$$\zeta(z_0) \sim \tfrac{1}{2}\sqrt{t_B}/|\mathbf{p}| \quad \text{or} \quad \tfrac{1}{2}\sqrt{u_B}/|\mathbf{p}|. \tag{1.61}$$

So for

$$l \geq 2|\mathbf{p}|/\sqrt{t_B} \quad \text{or} \quad 2|\mathbf{p}|/\sqrt{u_B} \tag{1.62}$$

$A_l(s)$ is exponentially small. This may be understood in physical terms: the range R of the force is given by $R^2 = t_B^{-1}$ or u_B^{-1}, and particles whose transverse separation or impact parameter b is greater than R are not scattered. Roughly speaking $l = b|\mathbf{p}|$.

The upper limit (1.62) on l applies at fixed s, but it may change if now s is allowed to be large. Then $\mathbf{p}^2 \sim \frac{1}{4}s$, so that at the bound-state pole $D_t(s,t(s,z_s')) \sim s^{-1}$, according to (1.58). It is possible, and indeed it is expected from the Regge theory described in the next chapter, that $D_t(s,t(s,z_s'))$ is larger for values of z_s' such that $t > 4m^2$ or $u < 4m^2$. In fact we expect it to be bounded by some power α of s, where α varies with t or u. This has the effect that for large l the dominant contribution to the Froissart-Gribov integral (1.54) will come from values of t and u such that s^α is as large as possible while still z_s' is close enough to ± 1 for the exponential in (1.55) not to provide too much damping. Hence instead of (1.59),

$$A_l(s) \sim l^{-\frac{1}{2}} \exp\Big(M(l+\tfrac{1}{2})/\sqrt{s} + \alpha \log(s/s_0)\Big) \tag{1.63}$$

where s_0 is some fixed scale and the value of M depends on the relevant range of values of t or u. The physical reason for this change is that, as we shall see in the next chapter, at high energy the force is not the result of the exchange of a single particle, but rather the simultaneous exchange of whole families of particles. Hence $A_l(s)$ will be exponentially small for

$$l \geq \alpha M^{-1}\sqrt{s}\ \log(s/s_0) \tag{1.64}$$

and the partial-wave series (1.45) may be truncated at this value. From (1.46), together with the unitarity constraint $0 \leq \eta_l \leq 1$,

$$\Big|A_l(s)\Big| = \Big|\frac{\eta_l e^{2i\delta_l} - 1}{2i\rho_s}\Big| \leq \frac{1}{\rho(s)} \tag{1.65}$$

and $\rho(s) \to 1$ as $s \to \infty$. Also, $|P_l(z_s)| \leq 1$. So, for large s

$$\begin{aligned} |A(s,t(z_s=1))| &\leq \sum_{l=0}^{l_{\rm MAX}} (2l+1) \\ l_{\rm MAX} &= \alpha M^{-1}\sqrt{s}\ \log(s/s_0). \end{aligned} \tag{1.66}$$

After the arithmetic progression is summed, this gives

$$|A(s,t(z_s=1))| \leq \text{constant} \times s\log^2(s/s_0). \tag{1.67}$$

Applying the optical theorem (1.25) then gives, when s is large,

$$\sigma^{\rm Tot}(s) \leq \text{constant} \times \log^2(s/s_0) \tag{1.68}$$

which is the Froissart bound[12].

Although the result (1.68) reproduces that of Froissart's original work[12] our derivation lacks its formal rigour. He assumed only that the dispersion relations require a finite number of subtractions and that the amplitude is polynomial bounded. From axiomatic field theory it proved possible to determine[13,14] the constant in (1.68):

$$\sigma^{\rm Tot}(s) \leq \frac{\pi}{m_\pi^2} \log^2(s/s_0) \tag{1.69}$$

although the scale s_0 remains unspecified. However if one chooses a reasonable hadronic scale, $s_0 \sim 1$ GeV^2 say, then the limit (1.69) is extremely high: 10 to 25 barns at Tevatron or LHC energies, that is in the range $\sqrt{s} = 1$ to 20 TeV.

A critical discussion of the formulation of asymptotic bounds in general and of their domain of validity can be found in [15].

1.8 The Pomeranchuk theorem

The Pomeranchuk theorem[16] asserts that, under certain quite strong assumptions, total cross sections for collisions of a particle and the corresponding antiparticle on the same target become asymptotically equal at high energy. For example, $\sigma^{\rm Tot}(\pi^+p)/\sigma^{\rm Tot}(\pi^-p) \to 1$ or $\sigma^{\rm Tot}(pp)/\sigma^{\rm Tot}(\bar{p}p) \to 1$ as $s \to \infty$.

As we are comparing particle and antiparticle interactions we are concerned explicitly with s-channel $\leftrightarrow$ u-channel crossing. From the optical theorem (1.25), to calculate the total cross sections we need the amplitude at $t = 0$. It is convenient to use the variable $\nu = P_1.P_2$, in terms of which, when $t = 0$,

$$\begin{aligned} s &= m_1^2 + m_2^2 + 2\nu \\ u &= m_1^2 + m_2^2 - 2\nu \end{aligned} \tag{1.70}$$

where m_1 and m_2 are the masses of the two particles. Thus the crossing simply takes $\nu \to -\nu$.

The forward scattering amplitude $A(\nu, t = 0)$ is analytic in the complex ν-plane cut along $(-\infty, -m_1 m_2)$ and $(m_1 m_2, \infty)$ with possible bound-state poles lying in the region $-m_1 m_2 < \nu < m_1 m_2$. For example in $\pi^\pm p$ scattering there will be poles at $\nu = \mp\frac{1}{2} m_\pi^2$ corresponding to the nucleon poles in the s- and u-channels.

For most physical scattering processes, amplitudes are neither symmetric nor antisymmetric under crossing. In general the process in the crossed channel is different from the one in the direct channel: the crossed channel for $\pi^+\pi^+$ scattering is $\pi^-\pi^+$ scattering; the crossed channel for π^+p scattering is π^-p or $\pi^+\bar{p}$ scattering; and the crossed channel for pp scattering is $\bar{p}p$ scattering. But we may construct amplitudes which are symmetric or antisymmetric under crossing. Take $\pi^+\pi^+$ and $\pi^-\pi^+$ scattering as an example and define

$$A_+(\nu, 0) = A(\pi^+\pi^+ \to \pi^+\pi^+) \qquad A_-(\nu, 0) = A(\pi^-\pi^+ \to \pi^-\pi^+). \tag{1.71}$$

Then the amplitudes which are symmetric and antisymmetric under crossing are given by

$$\begin{aligned} A^{\mathrm{S}}(\nu) &= \tfrac{1}{2}(A_+(\nu, 0) + A_-(\nu, 0)) \\ A^{\mathrm{A}}(\nu) &= \tfrac{1}{2}(A_+(\nu, 0) - A_-(\nu, 0)). \end{aligned} \tag{1.72}$$

We write fixed-t dispersion relations (1.38) for each of these. We introduce an integration variable ν', related linearly to s' and u' by equations similar to (1.70). For the symmetric amplitude the second integral in the dispersion relation for $A^{\mathrm{S}}(\nu)$ is obtained from the first by changing the sign of ν; also, according to (1.34) the discontinuity $D_s(s', t = 0, u')$ is just $2i \,\mathrm{Im}\, A^{\mathrm{S}}(\nu' + i\epsilon)$. For $A^{\mathrm{A}}(\nu)$ there are similar statements, except that we must in addition change the sign of the first integral to get the second one.

Because of the Froissart bound (1.69) the dispersion relations for the amplitudes $A_\pm$ or $A^{\mathrm{S,A}}$ require at most two subtractions. We introduce a fixed value ν_1, as in (1.43), and then each dispersion relation contains two subtraction constants, the values of the amplitudes at $\nu = \nu_1$, together with their derivatives at the same point. But if we choose ν_1 to be 0, the antisymmetric amplitude A^{A} vanishes there, as does the derivative of the symmetric amplitude A^{S}. Hence the dispersion relation for each of these amplitudes has only one subtraction constant:

$$\begin{aligned} \mathrm{Re}\, A^{\mathrm{S}}(\nu) - A^{\mathrm{S}}(0) &= \frac{\nu^2}{\pi} P \int_{m_\pi^2}^{\infty} d\nu' \frac{\mathrm{Im}\, A^{\mathrm{S}}(\nu' + i\epsilon)}{\nu'^2(\nu' - \nu)} \quad + \quad (\nu \to -\nu) \\ &= \frac{2\nu^2}{\pi} P \int_{m_\pi^2}^{\infty} d\nu' \frac{\mathrm{Im}\, A^{\mathrm{S}}(\nu' + i\epsilon)}{\nu'(\nu'^2 - \nu^2)} \end{aligned} \tag{1.73}$$

and

$$\begin{aligned} \mathrm{Re}\, A^{\mathrm{A}}(\nu) - \nu \frac{d}{d\nu} A^{\mathrm{A}}(0) &= \frac{\nu^2}{\pi} P \int_{m_\pi^2}^{\infty} d\nu' \frac{\mathrm{Im}\, A^{\mathrm{A}}(\nu' + i\epsilon)}{\nu'^2(\nu' - \nu)} \quad - \quad (\nu \to -\nu) \\ &= \frac{2\nu^3}{\pi} P \int_{m_\pi^2}^{\infty} d\nu' \frac{\mathrm{Im}\, A^{\mathrm{A}}(\nu' + i\epsilon)}{\nu'^2(\nu'^2 - \nu^2)} \,. \end{aligned} \tag{1.74}$$

As in (1.40), we have written just the real part of each dispersion relation. We recall that the amplitudes are real at $\nu = 0$.

At sufficiently high energy the optical theorem (1.25) gives

$$\operatorname{Im} A_{\pm}(\nu, 0) \sim 2\nu\, \sigma_{\pm}^{\mathrm{Tot}}(\nu). \tag{1.75}$$

Now we know from the Froissart bound (1.69) that $\sigma_{+}^{\mathrm{Tot}}$ and $\sigma_{-}^{\mathrm{Tot}}$ are both bounded by a constant times $(\log\nu)^2$ as $\nu \to \infty$. Suppose that

$$\sigma_{+}^{\mathrm{Tot}} - \sigma_{-}^{\mathrm{Tot}} \sim C\,(\log\nu)^n \qquad 0 < n \leq 2 \tag{1.76}$$

so that, from (1.75),

$$\operatorname{Im} A^{\mathrm{A}}(\nu) \sim 2C\nu\,(\log\nu)^n, \qquad 0 < n \leq 2. \tag{1.77}$$

Then (1.74) gives, for large ν,

$$\operatorname{Re} A^{\mathrm{A}}(\nu) \sim -\frac{4C\nu}{\pi(n+1)}(\log\nu)^{n+1}. \tag{1.78}$$

From (1.77), the real part of the antisymmetric amplitude $A^{\mathrm{A}}(\nu)$ exceeds the imaginary part by a factor $\log\nu$. This implies that the amplitudes become predominantly real at high energy. The original derivation of the Pomeranchuk theorem assumed that Re $A^{\mathrm{A}}(\nu) \to 0$ at high energy, so that $C = 0$ and

$$\sigma_{+}^{\mathrm{Tot}} - \sigma_{-}^{\mathrm{Tot}} \to 0 \tag{1.79}$$

at high energy. More refined derivations with weaker assumptions have obtained[17,18] the weaker condition

$$\sigma_{+}(s)/\sigma_{-}(s) \to 1 \tag{1.80}$$

as $s \to \infty$, but it is still necessary to assume that a limit exists. It has not been possible to prove from field theory that this should be true[15].